THE POLYNOMIAL NUMBER METHOD FOR COMPUTATION SIGNAL WAVEFORMS IN FREQUENCY DEPENDENT PARAMETER INTERCONNECTION LINES

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Abstract:

The paper concerns a computer-aided analysis of signal waveforms in interconnection lines. Due to the skin effect and the proximity effect, the resistance and inductance per unit length are frequency dependent. The Laplace transform of the voltage wave *U*(*s*) in such a line is a complicated function of *s*. For a time domain analysis the polynomial number method is proposed. We treat the polynomial numbers (PN) as a generalisation of the ordinary numbers of the decimal, binary etc. system. The PN rules of arithmetic operation, e.g. the floating-point arithmetic are simple and easy to the computer implementation.

1. INTRODUCTION

We observe different parasitic phenomena in interconnection media under high speed switching condition, in transmission line systems at high frequency operation etc. [1][8]. The skin and proximity effects are one of them and have influence on the waveform. The per unit-length impedance of a line depends strongly on frequency and causes impedance mismatches. We have examined and compared the results for a two-conductor cable and a concentric cable with series parameters to be independent of frequency $Z_C = \sqrt{(R + \mathrm{j}\omega L)/\mathrm{j}\omega C}$ and to be frequency dependent $Z_C = \sqrt{(R(f) + \mathrm{j}\omega L(f))/\mathrm{j}\omega C}$. Frequently the suitable relations are expressed by the Bessel functions. For transient analysis we take the Laplace transforms of line voltages and currents. The suitable impedance per of unit-length is the function of the complex variable *s*. For example, for a two-conductor cable the distributed series impedance can be expressed as follows

$$Z(s) = \frac{k}{\pi a\sigma}\left[\frac{I_0(ak)}{I_1(ak)} + 2\sum_{n=0}^{\infty}\left(\frac{a}{d}\right)^{2n}\frac{I_n(ak)}{nI_{n-1}(ak)}\right] + sL\,, \tag{1a}$$

where

k = $\sqrt{\mu\sigma}\sqrt{s}$

μ , σ - permeability and conductivity,

a - radius of conductors,

d - distance between the conductors,

$I_n(\,)$ - modified Bessel function of the first kind,

$L = \dfrac{\mu}{\pi}\ln\dfrac{d-a}{a}$.

The proximity effect influence on the external inductance is neglected. For a concentric cable we have:

$$Z(s) = \frac{k}{\pi a\sigma}\left[\frac{I_0(ak)}{aI_1(ak)} + \frac{K_0(bk)}{bK_1(bk)}\right] + sL \tag{1b}$$

where

a - radius of internal conductors,

b - internal radius of external conductors,

$K_n(\ )$ - modified Bessel function of the second kind,

$L = \dfrac{\mu}{2\pi} \ln \dfrac{b}{a}$.

As we know the Laplace transform of voltage in the cable (Fig.1) can be expressed by the travelling wave form:

$$U(x,s) = \frac{E_1 Z_1}{Z_C + Z_1}\left[\mathrm{e}^{-\gamma x} + q\,\mathrm{e}^{-\gamma(2l-x)}\right]\sum_{k=0}^{\infty} q_1^k q_2^k\, \mathrm{e}^{-2\gamma l k} \tag{2}$$

where

l - line length

Z - series impedance per unit length

Y - shunt admittance per unit length

$Z_\mathrm{C} = \sqrt{Z/Y}$, $\gamma = \sqrt{ZY}$, $q_i = \dfrac{Z_i - Z_\mathrm{C}}{Z_i + Z_\mathrm{C}}$, i = 1, 2 , ...

We assume $Y = sC$.

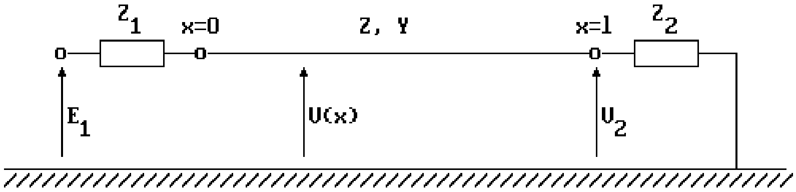

Fig. 1.

Characteristic parameters Z_C and γ as well as the reflection coefficients q_1, q_2 are functions of the impedance Z from eq. (1).

For the time-domain analysis we have to take the inverse Laplace transformation of eq. (2). This procedure is very difficult because of the complicated form of the transform.

2. THE POLYNOMIAL NUMBER METHOD

For such a kind of problems we elaborate a computer method denominated "the polynomial number method". The polynomial numbers (for convenience we will call PN) we treat as a generalisation of the ordinary numbers of decimal system. The PN system can also represent a large class of other positional number systems. The PN are defined as a set which is described by a sequence of elements a_n belonging to an arbitrary field with one sequence position to be pointed out (exactly PN are pairs ($\{a_n\}$, N) of sequences $\{a_n\}$ and natural numbers N). The PN we denote by

$\underline{a} = (\, a_{-N}\,\tilde{}\,a_{-N+1}\,\tilde{}\ldots\tilde{}\,a_{-1}\,\tilde{}\,a_0\,\tilde{},\, a_1\,\tilde{}\,a_2\,\tilde{}\ldots)$, (3)

N - natural number.

The PN method refers to the approach of numerical operators described by Bellert [2]. In eq. (3) a_n is a digit of PN and represents a traditional rational or complex number. The point marks the pointed out position ($a_0\,\tilde{}$,) - it is a radix point in positional number systems. It may be neglected, if a_1, a_2,...= 0. The sign "$\tilde{}$" serves to separation of the PN digits. If the digits a_n are also polynomial numbers then $\underline{a}$ is a polynomial number of the second order (^{2}PN).

The PN can be expressed in different equivalent forms:

$$\underline{a} = \sum_{n=-N}^{\infty} a_n\,(\,1\,\tilde{}\,0\,\tilde{}\,)^{-n} = \tag{4}$$

$$= (\,a_{-N}\,\tilde{},\, a_{-N+1}\,\tilde{}\; a_{-N+2}\,\tilde{}\ldots)(\,1\,\tilde{}\,0\,\tilde{}\,)^N = \tag{5}$$

$$= (\,a_{-N}\,\tilde{}\; a_{-N+1}\,\tilde{},\, a_{-N+2}\,\tilde{}\ldots)(\,1\,\tilde{}\,0\,\tilde{}\,)^{N-1}$$

The form of eq.(4) corresponds to a power series with p as a base i.e.

$$(\underline{a})_p = \sum_{n=-N}^{\infty} a_n p^{-n} = a_{-N} p^N + a_{-N+1} p^{N-1} + \ldots + a_{-1} p + a_0 + a_1 p^{-1} + \ldots = \ = (a_{-N}\,\tilde{}\,a_{-N+1}\,\tilde{}\ldots\tilde{}\,a_{-1}\,\tilde{}\,a_0\,\tilde{},\, a_1\,\tilde{}\ldots\tilde{}\,)_p \tag{6}$$

If $p = r = 2$ or 8 etc. and $a_n = 0, 1, ..., r\text{-}1$ we obtain the binary or octal etc. number system. If $p = 10$, $a_n = 0, ..., 9$ we notice a familiar rational number written in the positional decimal number system (eq.(6)). Then the separating signs and the radix are needless. Since $p = 1p + 0$ then in the PN notation there is p=($1\tilde{~}0\tilde{~})_p$. Hence eqs. (4) and (6) are equivalent.

Eq.(5) shows that the factor ($1\tilde{~}0\tilde{~}$) serves a role of the point shift operator. The variable p (or radix r) does not participate in any further arithmetic operations and is replaced by the PN factor ($1\tilde{~}0\tilde{~}$). The PN in a form of eq.(5) corresponds to the floating point arithmetic of rational numbers [1]. The PN rules of arithmetic operations are simple and easy to the computer implementation [4].

3. THE PN AND THE INVERSE LAPLACE TRANSFORMATION

A large class of $\boldsymbol{L}$-transforms which consists of rational functions or/and functions having their asymptotic expansions (e. g. Bessel functions) are possible to express as a PN. Eq.(2) belongs to such a class of functions.

After obtaining $F(s)$ in a form of the PN, we take the inverse $\boldsymbol{L}$-transformation to get the time domain solution. There are two approaches. First of them is connected with the inverse transformation of series [9]

$$f(t) = \boldsymbol{L}^{-1}\{ \sum_{n=0}^{\infty} a_n \, (s^{1/\nu})^{-n} \} \ , \ \nu = 1, 2, ... \tag{7}$$

In that case we make a replacement s = ($1\tilde{~}\ 0\tilde{~})^{\nu}$ for all components of the function $F(s)$. For further calculation in transform domain we use computer aided PN arithmetic. That causes all results -as well final result- are isomorphous with series (7).

The second one uses the analogy between the PN and the $\boldsymbol{Z}$-transform of f(t) [5]. This approach requires the replacement of s by constant polynomial number $\underline{p}$. The constant p is connected with numerical algorithm of differentiation. For example

$$\underline{p} = \frac{2}{h}\frac{(1\tilde{~} -1\tilde{~})}{(1\tilde{~}1\tilde{~})} = \frac{2}{h}(1\tilde{~},-2\tilde{~}2\tilde{~} -2\tilde{~}2\tilde{~}....)$$

in the case of the trapezoidal rule. All PN which are $\boldsymbol{Z}$-transforms of signals have got digits equal to samples of these signals in succeeding time steps h - as well $\boldsymbol{Z}$-transform of final result. We will not develop this problem now.

A brief explanation of the operation $\boldsymbol{L}^{-1}\{U(x,s)\}$ is necessary since the first approach was applied. In eq.(2) there are functions like $\exp(-\gamma l)$. Because of the form of Bessel function argument, which we find in γ, it is useful to take series (7) with ν=2 for our calculating procedure. In this case $s = (1\tilde{~}0\tilde{~}0)$ and $\sqrt{s} = (1\tilde{~}0\tilde{~})$. As the final result for $U(x,s)$ in the PN field we get a sum of terms like

$$\underline{A} = \underline{f}\exp(-\gamma\alpha) \tag{8}$$

where $\underline{f}$ is PN which corresponds with any $f(t)$ function (i.e. we can calculate inverse transform of $\underline{f}$), α is real number. But we cannot compute $\exp(-\gamma\alpha)$. For example series

$$\sum_{n=0}^{\infty} (-\gamma\alpha)^n / n! \tag{9}$$

is not convergent in the PN space. We have to split argument of exp function ([7] pp.56-58)

$$-\gamma\alpha = -(\gamma_{-2}\tilde{~}\gamma_{-1}\tilde{~}\gamma_0\tilde{~},\gamma_1\tilde{~}\gamma_2\tilde{~}...\tilde{~})\ \alpha =$$

$$= -\gamma_{-2}\alpha\, s - \gamma_{-1}\alpha\,\sqrt{s} - (\gamma_0\tilde{~},\gamma_1\tilde{~}\gamma_2\tilde{~}...\tilde{~})\alpha$$

where s = ($1\tilde{~}0\tilde{~}0\tilde{~}$) , $\sqrt{s}$ = ($1\tilde{~}0\tilde{~}$). Let

$$\tau = \gamma_{-2}\alpha\ ,\ \beta = \gamma_{-1}\alpha\ ,\ \underline{g} = (\gamma_0\tilde{~},\gamma_1\tilde{~}\gamma_2\tilde{~}...\tilde{~})\alpha\ .$$

Now term $\underline{A}$ (8) assumes a form

$$\underline{A} = e^{-\tau s}\, e^{-\beta\sqrt{s}}\, e^{-\underline{g}}\, \underline{f} \tag{10}$$

Function $e^{-\tau s}$ represents the known translation operator. There is possible to calculate $e^{-\underline{g}}$ (series $\Sigma\,(-\underline{g})^n / n!$ is convergent), therefore we obtain PN $\underline{F} = e^{-\underline{g}}\,\underline{f}$. (For calculate PN exponential function we have far faster algorithm then mentioned series [6]). The only one problem is inverse transformation $\Phi(t) = \boldsymbol{L}^{-1}\{\, e^{-\beta\sqrt{s}}\,\underline{F}\}$. After obtaining $\Phi(t)$ we have $A(t) = \Phi(t-\tau)$.

$\underline{F}$ is in form

$$\underline{F} = (\,0\tilde{},\, F_1\tilde{}F_2\tilde{}F_3\tilde{}\ldots) = F_1\sqrt{s}^{\,-1} + F_2\sqrt{s}^{\,-2} + F_3\sqrt{s}^{\,-3} + \ldots$$

(digits F_0, F_{-1}, F_{-2}... of PN $\underline{F}$ before radix point are equal to zero because digits of $\underline{E}_1$ before radix point are equal to zero). Thus

$$\Phi(t) = \boldsymbol{L}^{-1}\{\, e^{\beta\sqrt{s}} \sum_{n=1}^{\infty} F_n \sqrt{s}^{\,-n}\,\} = \sum_{n=1}^{\infty} F_n y_n(t;\beta) \tag{11}$$

where

$$y_0(t;\beta) = \frac{1}{\sqrt{\pi t}}\exp\left(\frac{-\beta^2}{4t}\right),\ \ y_1(t;\beta) = \operatorname{cerf}\left(\frac{\beta}{2\sqrt{t}}\right) \tag{12}$$
$$y_{n+1}(t;\beta) = (2t y_{n-1}(t;\beta) - \beta y_n(t;\beta))/n$$

Series (11) we obtain for ν=2 in (7). In simple case of $z(s)$ e.g. $z(s)=sL+R$, where L, R are frequency independent, we can take ν=1. In this case we obtain Taylor series instead of series (11).

4. NUMERICAL RESULTS

In this way we have found the signal waveform at the output end of the lines (U_2) as well as the reflected wave (U_2^-). In all figures the solid lines show the waveforms in the frequency dependent parameter. The dashed lines are obtained at the assumption that the parameters are frequency independent.

For the two-conductor line "o--o" we assume a = 0.45 mm, d = 7.8 mm, ε = 1.25 ε_0, σ = 56 10^6 S/m, $\mu = \mu_0$; $Z_2 = R_2 = \sqrt{L/C}$ = 299.8 Ω, (L = 1.1176 µH/m, C = 12,431 pF/m, R = 50 mΩ/m).

For the concentric line "(o)" we assume a = 1.0 mm, b = 3.55 mm, ε = 2.25 ε_0, σ = 56 10^6 S/m, $\mu = \mu_0$; $Z_2 = R_2 = \sqrt{L/C}$ = 101.36 Ω, (L = 0.50678 µH/m, C = 49.331 pF/m, R = 5,06 mΩ/m).

In the both cases we take $Z_1 = 0$, $e_1 = 1(t) - 1(t-t_0)$, $t_0 < \sqrt{LC} = \tau$, $t_0 = 0{,}1\tau$.

(o)

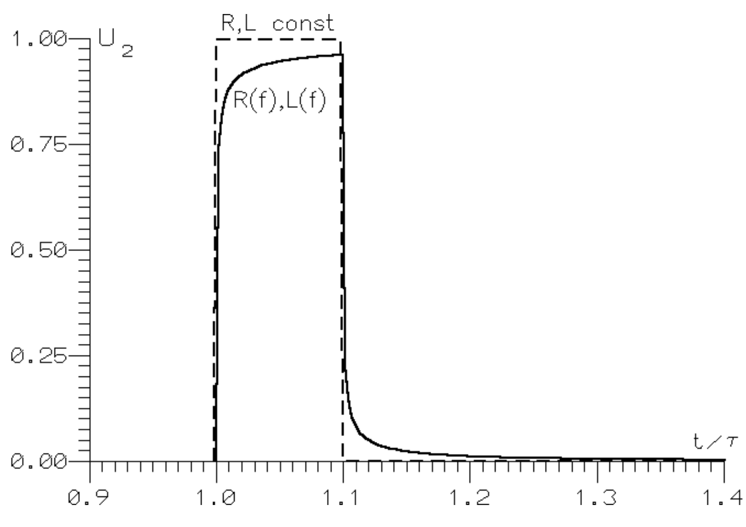

Fig.2. Output voltage waveforms. t_0 = 0.1 τ

o—o

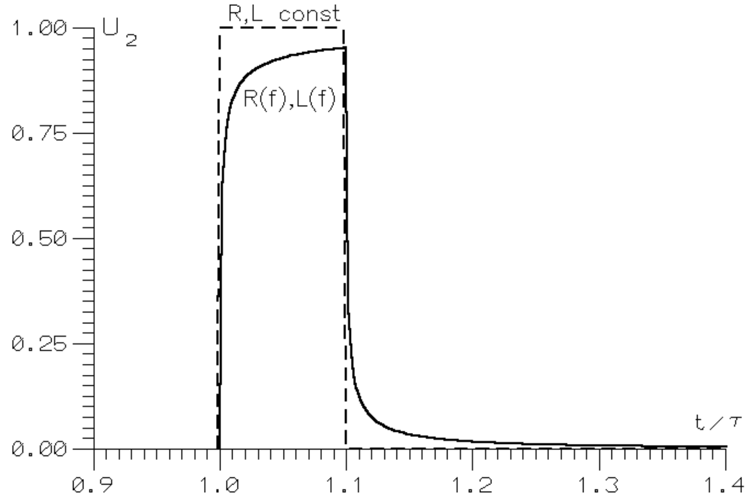

Fig.3. Output voltage waveforms. t_0=0.1 τ

(o)

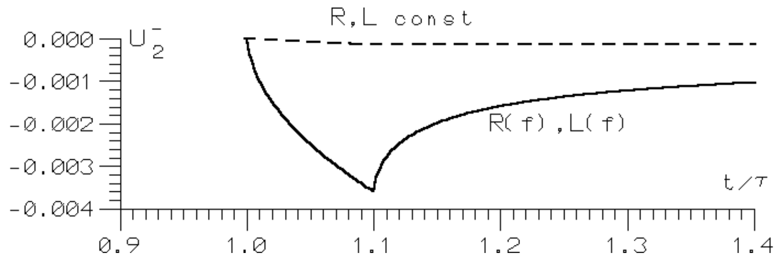

Fig.4. Reflected waves at output-end. t_0=0.1 τ

o—o

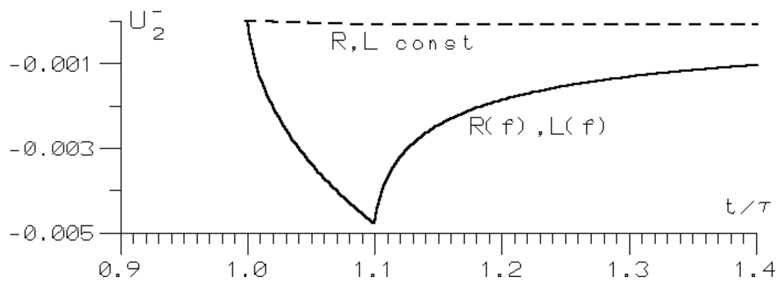

Fig.5. Reflected waves at output-end. t_0=0.1 τ

(o)

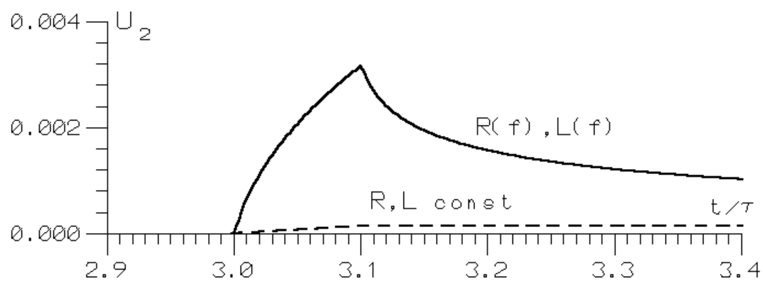

Fig.6. Reflected waves coming back to the output-end. t_0=0.1 τ

o—o

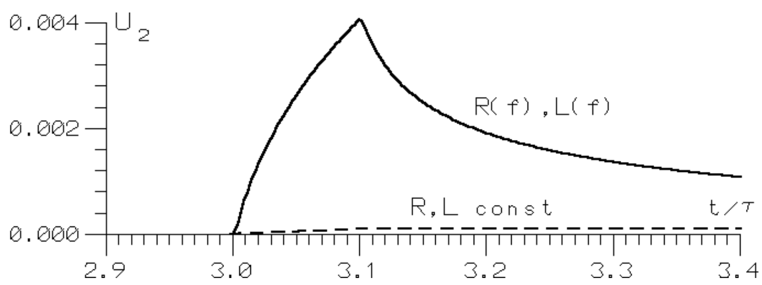

Fig.7. Reflected waves coming back to the output-end. t_0=0.1 τ

(o)

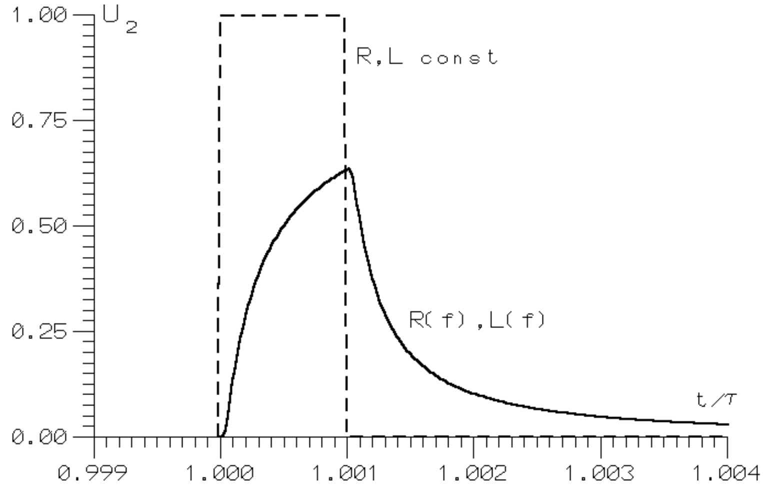

Fig.8. Output voltage waveforms. t_0=0.001 τ

o—o

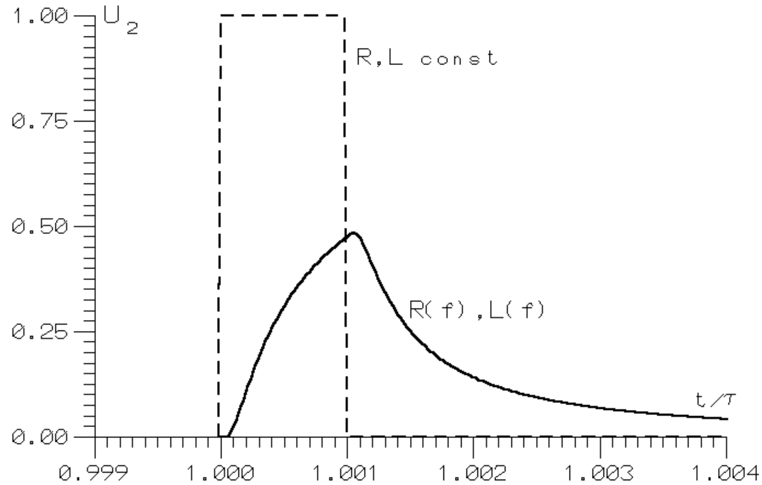

Fig.9. Output voltage waveforms. t_0=0.001 τ

5. FINAL REMARKS

The skin and proximity effects have considerable influence on the waveform, particularly for wide spectrum signals. If the source E_1 generates the pulse-sequence, then arisen reflected waves superpose giving some level of noise. The intensity of these effects is greater in the two-conductor lines. The PN method is very effective in such an investigation.

The accuracy of results depends on three factors. The influence has the mathematical model error, the PN method error and the round-off error. The PN method error is connected with the PN truncation during the computation procedure and we can estimate it influence on final results.

6. REFERENCES

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