Discrete representations of generalized time domain function in symbolic-numerical analysis

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Abstract – **Computer simulation sometimes leads to the results, which can be supposed as the numerical instability. In this paper it is shown, that some of that results should be accepted – they are generalized time domain functions in Mikusinski's operational calculus, and after some operations they can be useful solutions. The properties of discrete representations of these generalized functions are demonstrated using repetitive symbolic-numerical analysis and examples of applications are discussed.**

1 INTRODUCTION

Computer analysis of signals of continuous time is based usually on sampling process. This process is applied to bandlimited signals which satisfies the sampling theorem assumption. In this paper discrete representations of abstract, generalized, not bandlimited functions are discussed. The series of samples of such functions have particular properties, which can be discovered in the repetitive analysis of the symbolic-numerical mathematical models. An example of generalized function is derivative of non-smooth function or function containing negative delay operator in case of non-negative domain of function argument. Although discrete representations of such functions are problematic, they can be utilized to discover mathematical model property or to improve final result of analysis selecting proper order of calculation in sequence of expressions.

The theory of generalized time domain function was developed by Jan Mikusinski in monograph "Operational calculus" [1, 2, 3], which was formally correct explanation of heuristic Heaviside's operational calculus. The conclusions of Mikusinski's operational calculus are close to the Laplace transform theory, but what is important Mikusinski's operational calculus introduces generalized time domain functions without time domain to complex domain functions mapping. In his operational calculus the time domain functions cooperate with operators like integration, differentiation or shifting, giving new generalized, abstract time domain function.

We can consider discrete equivalence of this calculus, where sequence of samples corresponds to continuous function, and discrete algorithms of integration, differentiation or shifting correspond to operators. The results of these operations are also sequences of the samples, which correspond to abstract functions, like n-th derivative of Dirac's delta function $\delta^{(n)}(t)$, or function with negative delay operator, and we can investigate their properties.

The investigation is a purpose of the paper. The property of discrete representation of generalized function is presented using symbolic numerical analysis with polynomial number algebra [4, 5], and is illustrated using animated graphs in repetitive analysis process [6] – in the author's opinion only dynamic, animated graphs can expose specific quality of the sequences of samples of generalized functions.

2 GENERALIZED TIME DOMAIN FUNCTION SAMPLING

Let us consider computer modelling problem where objective is continuous time function based on samples processing utilizing discrete models of systems. Let us establish notations used in this paper. After Mikusinski [1, 2] symbol $\{f(t)\}$ will denote a whole function f to make distinction between a symbol $f(t)$, which denotes value of the function at point t, so $f \equiv \{f(t)\}$. For example, $\{1\}$ means constant function with value 1 at every point t. $\{f_0, f_1, ...\}$ and $\{f_k\}$ will denote sequence of values f_k. Let us denote sequences using underlined letters: $\underline{f} \equiv \{f_k\}$.

Let us assume, that our task is to compute continuous time function $\{f(t)\}$. Using some numerical algorithm we can determine sequence of coefficients $\{f_k\}$, which can be coefficients of function series or sequence of the samples which allow to restore $\{f(t)\}$. For example, we can determine sequence $\{f(0), f(h), f(2h), f(3h), ...\}$, where $h > 0$ denotes sampling interval, and after linear interpolation we can restore continuous time function. Generally the reconstruction of continuous time function is possible, when an operation of approximation $A_{pprox}()$ is predetermined: $\{f_k\} \mapsto \{A_{pprox}(\{f_k\},\ t)\}$. We will say that sequence $\{f_k\}$ approximates function $\{f(t)\}$, what will be denoted $\{f_k\} \approx \{f(t)\}$, when $\{A_{pprox}(\{f_k\}, t)\} \approx \{f(t)\}$.

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Shortly it will be also denoted: $\underline{f} \approx f$, when $A_{pprox}(\underline{f}) \approx f$.

Objective function f is usually unknown, and to ensure that approximation based on samples sequence is accurate enough we can find solution decreasing sampling interval. Thus we have sequence $\underline{f}_i$ of samples sequences, which corresponds to sequence of continuous time function $A_{pprox}(\underline{f}_i)$, and limit of this sequence is objective function f : $f = \lim A_{pprox}(\underline{f}_i)$

The crucial observation regarding generalized function is, that sequence of function $A_{pprox}(\underline{f}_i)$ is not convergent. So decreasing sampling interval a sequence of different functions $A_{pprox}(\underline{f}_i)$ is obtained, which seems as if not proper simulation result. But it could be proper discrete representation of generalized function, and in some cases we can utilize it to retrieve information regarding modelled system. The examples will be shown in the next sections. These examples demonstrate divergent sequences of functions $A_{pprox}(\underline{f}_i)$. In the paper only few elements of these sequences can be presented. Some better information regarding divergent property of these sequences can be retrieved using animated graphs in repetitive symbolic-numerical analysis. These graphs are presented on the web page: http://www.pei.prz.rzeszow.pl/~kubaszek/smacd06/

The graphs of generalized functions restored from their discrete representation often look strange and negate our intuition. But these functions are abstractions. What is important (and what was is underlined in Mikusinski monograph) after some transformation these functions return to space of non-generalized functions.

3 DISCRETE EQUIVALENCE OF GENERALIZED FUNCTION

3.1 Polynomial number calculator

The time domain symbolic-numerical analysis presented in the next sections is based on Z-transform and polynomial number calculator (i.e. library of computer algorithms). The sequence of samples $\{f_0, f_1, f_2, ...\}$ corresponds one-to-one to Z-transform $F(z) = f_0 z^0 + f_1 z^{-1} + f_2 z^{-2} + ...$, and calculation in Z-transform domain can be performed with the polynomial number calculator. The polynomial number corresponding to sequence $\{f_0, f_1, f_2, ...\}$ is generalized number with digits values equal to samples values: $(\tilde{}\, f_0 \,\tilde{}, f_1 \,\tilde{}\, f_2 \,\tilde{}\, ...\tilde{})$, and what is important, the polynomial number method is computer aided operational calculus. All PN operations are easy for computer implementation – four fundamental operations +, -, *, /, and functions like power function to real exponent or exponential function. These functions will be utilised in the examples.

The functions of continuous time will be defined for $t \geq 0$, and will be in form of functions of Heaviside's operator p [7, 8]

$$p = \frac{1}{\int_0^t} \tag{1}$$

with a separated factor $p\,\{1\}$, thus remainder part of these functions denote Laplace transform formulas $F(p)$ corresponding to $\{f(t)\}$:

$$\{f(t)\} = F(p)\; p\; \{1\} \tag{2}$$

where multiplication of whole functions means theirs convolution.

The Laplace transform $F(s)$ is the function with complex argument s, but a form of function $F(s)$ is identical to form of function $F(p)$ in Mikusinski's operational calculus. The important advantage of Mikusinski's operational calculus is that the equation (2) is equality in generalized time domain space, not a mapping into different domain.

The representation of continuous time function in form (2) could be convenient for readers being used to Laplace transform. For example, constant function $\{1\}$, that is function with value 1 for all $t \geq 0$, satisfies equation:

$$\{1\} = (1 / p)\;\; p\; \{1\} \tag{3}$$

thus $F(p) = 1 / p$. Another example is

$$\{\sin(t)\} \;= (1 / (p^2 + 1))\;\; p\; \{1\} \tag{4}$$

with $F(p) = (1 / (p^2 + 1))$. Using rational functions of Heaviside's operator p and delay operators $\exp(-\,T\,p)$ a wide range of continuous functions and functions with simple (jump) discontinuity can be expressed in form (2).

Expression (2) can denote not only continuous function, but also generalized, abstract function in Mikusinski's operational calculus. For example, formula

$$p^n\;\; p\; \{1\} \tag{5}$$

where $n \geq 0$ denotes abstract function: n-th derivative of Dirac's delta function $\delta^{(n)}(t)$.

The equation (2) can be treated as defined in expression $F(p)\,p$ algorithm of deformations of constant function $\{1\}$, which contains operators of integration, differentiation or delay. Replacing continuous time function $\{1\}$ in (2) by sequence of samples $(\tilde{}\,0.5\,\tilde{}, 1\,\tilde{}\,1\,\tilde{}\,1\,\tilde{}...\tilde{})$, and replacing Heaviside's operator (1) utilizing algorithm of numerical integration we obtain expression for approximate sequence of samples of $\{f(t)\}$ function:

$$\underline{f} = F(\underline{p})\; \underline{p}\; (\tilde{}\,0.5\,\tilde{}, 1\,\tilde{}\,1\,\tilde{}\,1\,\tilde{}...\tilde{}) \tag{6}$$

where $A_{pprox}(\underline{f}) \approx f$, i.e.

$$A_{pprox}(\sim f_0 \sim, f_1 \sim f_2 \sim \ldots \sim) \approx \{f(t)\} \quad \text{and}$$

$$f_0 \approx f(0)\,,\; f_1 \approx f(1h)\,,\; f_2 \approx f(2h)\,, \ldots$$

The polynomial number $\underline{p}$ based on trapezoidal rule has form:

$$\underline{p} = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\bigg|_{z=(\sim 1 \sim 0 \sim)} = \frac{2}{h}\frac{(\sim 1\sim, -1\sim)}{(\sim 1\sim,\ 1\sim)} \qquad (7)$$

$$= \frac{2}{h}(\sim 1\sim, -2\sim 2\sim -2\sim 2\sim \ldots)$$

where h is the sampling interval. Decreasing this interval we get sequence of approximations $A_{pprox}(\underline{f})$ of continuous time function which can converge to f function. The objective of this paper is to show the numerical experiments in cases, when formula $F(p)\, p\, \{1\}$ denote abstract, generalized function and sequence of approximations $A_{pprox}(\underline{f})$ is divergent.

3.2 Example with high order derivative

Let us consider function a which Laplace transform is $F(p) = p$:

$$a = p\;\; p\,\{1\} \qquad (8)$$

This abstract function is derivative of Dirac's delta function $\delta^{(1)}(t)$. Applying $F()$ to (6) we obtain series of samples:

$$\underline{a} = \underline{p}\cdot\underline{p}\cdot(\sim 0.5\sim, 1\sim 1\sim 1\sim \ldots\sim)$$

$$= \underline{p}^2 \cdot \frac{1}{2}\frac{(\sim 1\sim,\ 1\sim)}{(\sim 1\sim, -1\sim)}$$

$$= \left(\frac{2}{h}\frac{(\sim 1\sim, -1\sim)}{(\sim 1\sim,\ 1\sim)}\right)^2 \cdot \frac{1}{2}\frac{(\sim 1\sim,\ 1\sim)}{(\sim 1\sim, -1\sim)} \qquad (9)$$

$$= \frac{2}{h^2}(\sim 1\sim, -2\sim 2\sim -2\sim 2\sim \ldots\sim)$$

Linear interpolation of sequence of samples (9) gives continuous time approximation $A_{pprox}(\underline{f})$ which is not convergent when sampling interval h is decreased. Fig. 1 shows approximations with different interval h.

Usefulness of such abstract functions consist in their processing in physical systems to non-abstract functions. For example, double integration of function a, i.e. $p^{-2}\, a$ gives non-abstract jump function. This process can be also observed applying discrete version of operator p^{-2}

$$\underline{b} = \underline{p}^{-2}\;\underline{a} = (\sim 0.5\sim, 1\sim 1\sim 1\sim \ldots\sim) \qquad (10)$$

Fig. 2 illustrates that approximations based on $\underline{b}$ are convergent by decreasing interval h.

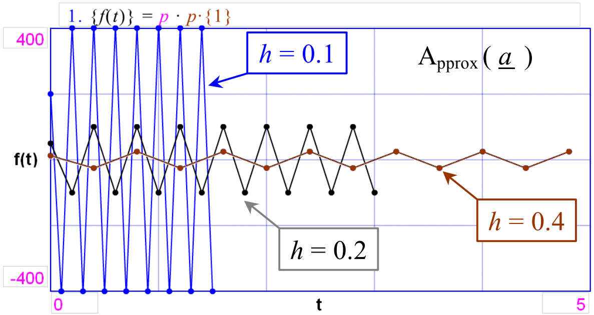

Figure 1. Approximations of function $\{a(t)\} = p\;\; p\;\{1\}$ $= \{\,\delta^{(1)}(t)\}$ based on their discrete representation.

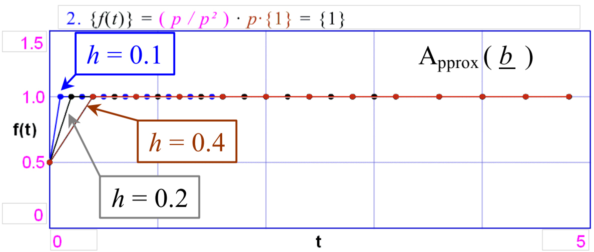

Figure 2. Approximations of function $\{b(t)\} = \{1\}$

3.3 Example with negative delay operator

Function with negative delay operator is abstraction in domain of function defined for $t \geq 0$ (see Mikusinski's remarks in [2] on the page 135). This function can appear in circuits with transmission lines, when ideal delay is extracted as shifting of samples, as it was shown in [9]. The approximations in fig. 3 show linear interpolations of discrete representations of function

$$x = \frac{p\cdot\{1\}}{p^2+p+4}\exp(-T_0\sqrt{p^2+1}) \qquad (11)$$

where $T_0 = -\,0.1$ introduces negative delay.

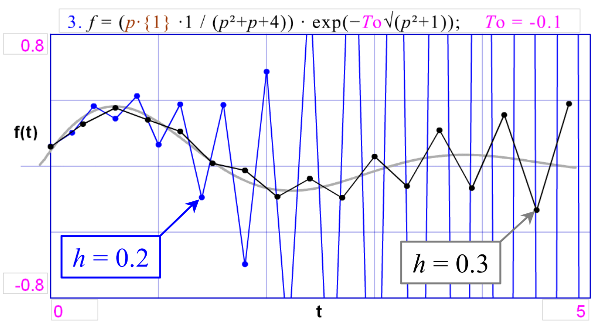

Figure 3. Discrete representation of function containing negative delay operator.

Approximations obtained applying formula (11) to (6) are not convergent when sampling interval h is decreased. But these series of samples are proper discrete representations of a function with negative delay operator – the function which is marked in the fig. 3 with grey smooth line. This is proper discrete representation (although strange) in form of series of samples defined for non-negative discrete time: $\underline{x} \approx (\tilde{}\ x(0)\ \tilde{},\ x(1h)\ \tilde{}\ x(2h)\ \tilde{}\ \ldots\tilde{})$. After applying operator which neutralize negative delay, sequence of approximation is convergent, as it was shown in the fig. 4 for function

$$y = \frac{p \cdot \{1\}}{p^2 + p + 4} \exp(-T_0 \sqrt{p^2 + 1} - (-T_0)p) \quad (12)$$

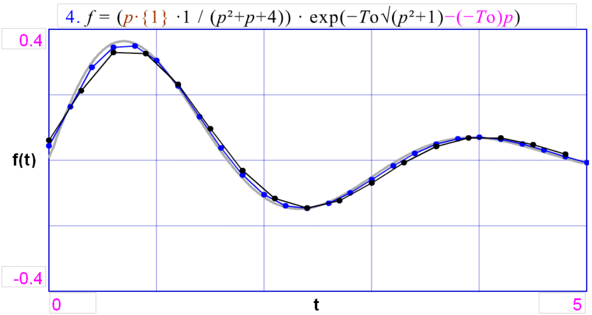

Figure 4. Neutralization of negative delay operator.

The value of delay $T_0 = -0.1$ in this example is smaller than sampling interval $h = 0.3$ or $h = 0.2$, so even improper way of rounding pure delay to multiply of sampling interval can cause abstract function effects as shown in fig. 3.

4 DISADVANTAGES OF DISCRETE REPRESENTATION OF GENERALIZED FUNCTION

4.1 Divergence

It is difficult to retrieve useful information from divergent sequence of approximations. To discover a kind of generalized function we have to integrate it few times or apply delay operator.

4.2 Floating point overflow

The negative aspect of discrete representation of generalized function is divergence of absolute values of samples to infinity. So we should avoid this representation in intermediate result. For example, formula (12) and (13) means the same function with neutralized negative delay operator ($T_0 < 0$), but discrete representation of part (13a) of the formula (13) contains abstract function, and leads to very high absolute values of samples, which are neutralized in part (13b), only if overflow does not occur.

$$\frac{p \cdot \{1\}}{p^2 + p + 4} \exp(-T_0 \sqrt{p^2 + 1}) \cdot \quad (13a)$$

$$\cdot \exp(-(-T_0)p) \quad (13b)$$

The final result of approximation using (13) is shown in fig. 5. In some conditions floating point overflow damages samples values. In the sequence of samples calculated using formula (12) this effect is not observed.

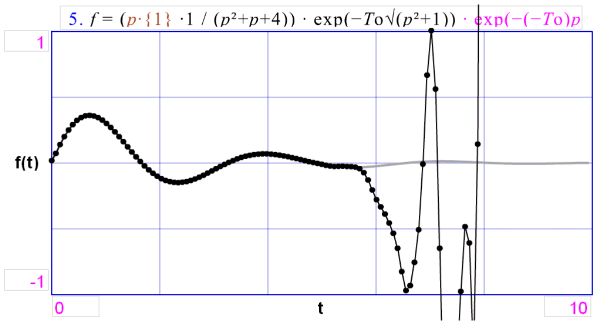

Figure 5. Discrete representation of function containing negative delay operator as intermediate result. $T_0 = -0.1$; $h = 0.08$.

5 UTILIZATION OF GENERALIZED FUNCTION REPRESENTATION PROPERTIES

The sequence of samples corresponding to generalized function has special properties. One of them is, that samples significantly change values at each sampling step. Another property is divergence of approximations by decreasing sampling interval. This knowledge can be used to filter characteristic of DSP algorithm closed in black box. This black box can be hardware signal processor, but also newly programmed DSP algorithm, which should be tested or debugged. When we have possibility to change sampling interval we can obtain convergence or divergence in sequence results. The idea is that we can observe output of black box applying different kind of generalized functions at input.

5.1 Discovering order of integration of DSP black box

We can test hypothesis that DSP black box simulates an analog system with R-th order of integration applying at the input digital representation of function $\delta^{(n)}(t)$

$$\underline{x}_n = \underline{p}^n \ \underline{p}\,(\tilde{}\,0.5\tilde{}, 1\tilde{}\,1\tilde{}\,1\tilde{}\ldots\tilde{}) \qquad (14)$$

which for $n \geq 0$ is the generalised function, for $n = -1$ is the step function, and for $n < -1$ is the continuous function. Starting from arbitrary value of n we should determine such value of $n = R$, that output signal corresponding to $\underline{x}_R$ is non-generalised function and corresponding to $\underline{x}_{R+1}$ is generalised function. Fig. 6 presents responses of black box algorithm which Z transfer function is digital equivalence of 2-nd order continuous transfer function $1 / (p^2+p+4)$:

$$\underline{T} = \frac{1}{\left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} + 4} \qquad (15)$$

The responses for $n = 0$, 1, 2 are convergent when decreasing sampling interval h. The fig. 6 shows only one case of step interval. To perform experiments with convergence we should use interactive graphs in repetitive symbolic-numerical analysis. The response when $n = 3$ is divergent. The conclusion is that black box simulates system with integration of 2-nd order.

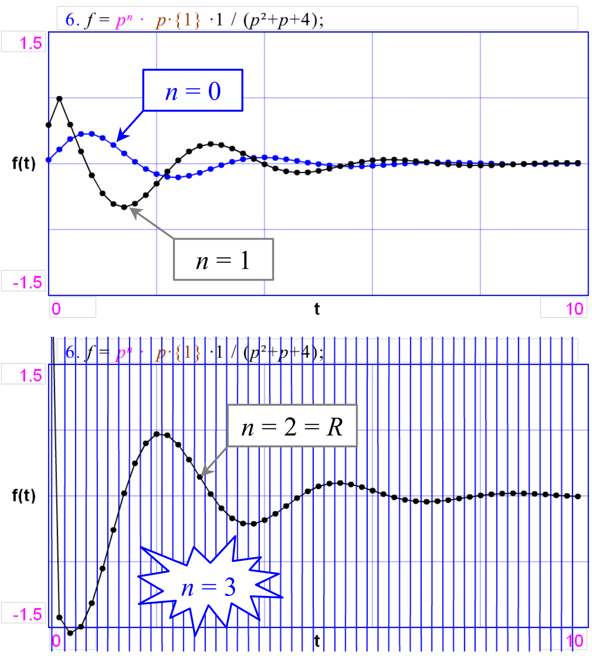

Figure 6. Signals at DSP black box output, when input is digital representation of function $p^n \ \ p\ \{1\}$.

The polynomial number calculator is able to evaluate the power function with real number exponent. So it is interesting, that we can also test DSP black box using input signal (14) with non-integer n. The example is shown in fig. 7. The response in case of $n = 2.3$ case is evidently divergent, in case of $n = 2.0$ is convergent, but in case of $n = 1.9$ divergence is also observed (perhaps important value of n is the highest by which convergence is observed). The function p^n with non-integer n is as if non-integer order of derivation. The power functions of operators with real exponent were discussed by Mikusinski in his monographs [1, 2].

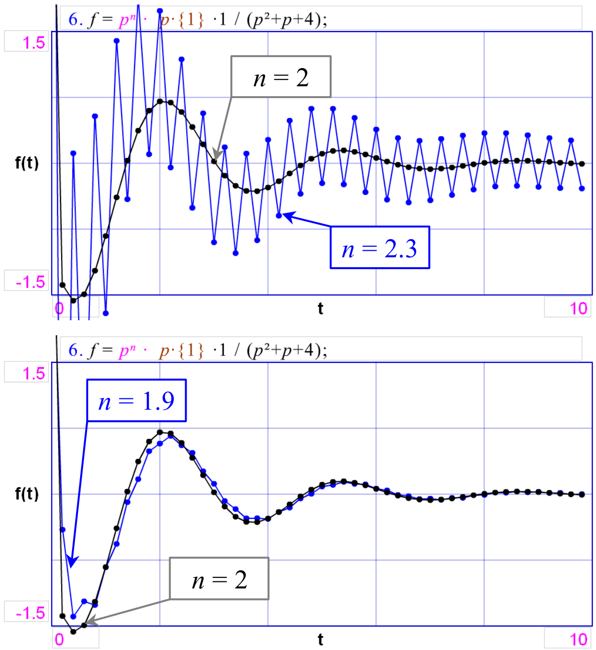

Figure 7. DSP black box response, when input is digital representation of $p^n \ \ p\ \{1\}$, and n is non-integer value.

5.2 Discovering delay in DSP black box

Hypothesis that DSP black box contains delay operator can be tested processing signal at input with digital equivalence of function with negative delay operator $\exp(-\,a\,p)$, where $a < 0$. In repetitive analysis process a limit value a can be determined – the limit when black box response becomes generalized function. Let us consider DSP black box with transfer functions

$$\underline{K} = \frac{1}{\underline{p}^2 + \underline{p} + 4}\exp(-2\sqrt{\underline{p}^2 + 1}) \qquad (16)$$

where $\underline{p}$ is defined by (7). Impulse response for this transfer function is shown in fig. 8. Digital transfer function is equivalence of continuous transfer function which inserts delay $T_0 = 2$; but impulse response approximation shown in fig. 8 does not point this value distinctly.

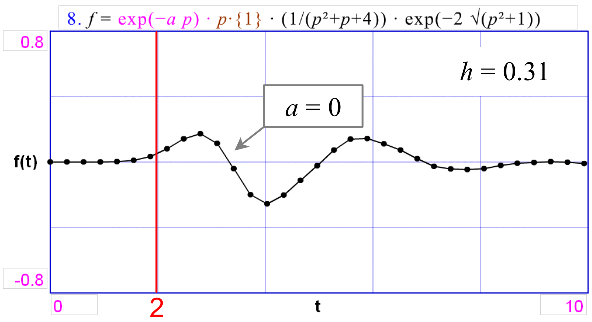

Figure 8. Impulse response of DSP black box containing delay operator.

In repetitive symbolic-numerical analysis we can notice (see fig. 9), that applying a signal containing discrete equivalence of negative delay operator ($a < 0$)

$$\exp(-a\,\underline{p}) \qquad (17)$$

to the DSP black box – this signal is calculated as polynomial number exponential function result – exists limit value $a = -2.05$, below which responses are convergent to non-generalized function. This way we can discover DSP black box delay value with accuracy higher than sampling interval.

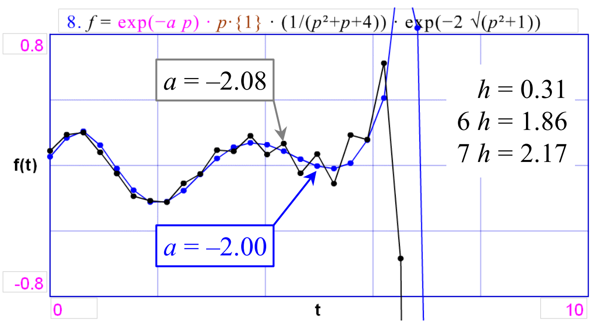

Figure 9. Detection of delay operator value using repetitive symbolic-numerical analysis.

6 CONCLUSIONS

Generalized time domain functions, like derivative of non-smooth function or function containing negative delay operator are used in modelling of continuous time system. In the paper discrete representations of such abstract functions were explored. Although series of samples of these functions pretend numerical instability case, although sampling theorem seems to be broken, although approximations of these functions are not convergent to any continuous time function, discrete equivalences of such functions act the same way as in continuous time domain. Knowledge regarding special properties of sequence of samples of such functions can be utilized to test or to improve DSP models. Repetitive symbolic-numerical analysis which take into account generalized functions illustrates abstract functions introduced 50 years ago by Mikusinski.

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Note: Full text versions of some mentioned above papers are available as electronic file at http://www.pei.prz.rzeszow.pl/~kubaszek . Interactive graphs utilizing repetitive symbolic-numerical analysis can be tested at sub page: ~kubaszek/smacd06/