POLYNOMIAL NUMBER METHOD - COMPUTER IMPLEMENTATION OF SOME OPERATIONAL CALCULI

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Abstract. The paper presents a polynomial number method, in which generalised numbers are used to analyse transient signals, especially in systems described by non-rational transfer functions. Connections between polynomial numbers and operators over continuous or discrete signals are discussed.

Key Words. Algebraic operational calculus, Laplace and Z transform, non-rational transfer function.

1. INTRODUCTION

A polynomial number method refers to the approach of numerical operators described by Bellert [1]. A polynomial number (PN) is a generalised number which can correspond one to one to a function $f(t)$ or to a sequence $\{f_k\}$ (table 1). The Polynomial number method is a computer aided operational calculus. As well PN operations as PN functions have easy computer implementation.

Table 1. Signals and numbers

Type of signal	***Equivalent number***	***Computer implementation***
DC	real number	real number (*N* bits)
AC	complex number	2 real numbers
transient	polynomial number	*M* real numbers and integer number

2. POLYNOMIAL NUMBER

2.1. Definition

A polynomial number (PN) can be treated as a generalisation of the ordinary real number of a decimal system. The digits of the PN are elements of any arbitrary field, for example, a real number field. The polynomial number can be described as a sequence of elements a_n belonging to any arbitrary field with one sequence position to be pointed out. That means the PN is a pair $(\{a_n\}, N)$ of a sequence $\{a_n\}$ and natural number $N < \infty$ [7]. The PN will be denoted:

$$\underline{a} = (\tilde{}\ a_{-N}\ \tilde{}a_{-N+1}\tilde{}\ldots\tilde{}a_{-1}\tilde{}a_0\tilde{},\ a_1\tilde{}a_2\tilde{}\ldots\tilde{}) \tag{1}$$

where a_n will be called PN "digits" ($n = -N, -N+1, \ldots \infty$), sign " ˜ " is used to separate the "digits" and " ˜, " distinguishes special position in the series of "digits". It is possible to concatenate any, but finite, number of the elements equal to zero to the left side of the sequence of "digits". If $a_i = a_{i+1} = a_{i+2} = \ldots = 0$ then all can be omitted in PN notation. If " ˜, " is omitted then $a_1 = a_2 = a_3 = \ldots = 0$. 0 denotes a neutral element of addition and 1 denotes a neutral element of multiplication in the "digits" field.

2.2. PN operations

$$\underline{a} = \underline{b} \Leftrightarrow a_j = b_j;\ \ j = -N,\ldots,0,1,\ldots \tag{2}$$

$$\underline{a} + \underline{b} = (\tilde{}\ a_{-N} + b_{-N}\tilde{}\ldots\tilde{}\ a_0 + b_0\tilde{},\ a_1 + b_1\tilde{}\ldots\tilde{})$$

where $\underline{a} = (\tilde{}\ a_{-N}\tilde{}\ldots\tilde{}\ a_0\tilde{},\ a_1\tilde{}\ldots\tilde{})$ and $\underline{b} = (\tilde{}\ b_{-N}\tilde{}\ldots\tilde{}\ b_0\tilde{},\ b_1\tilde{}\ldots\tilde{})$. It is not assumed here that $a_{-N} \neq 0$ or $b_{-N} \neq 0$.

Example 2.1 ("digits" are real numbers)
(˜ 1.8˜, -2˜ 0˜) + (˜ 25˜ 1.8˜, 7.6˜)=(˜ 25˜ 3.6˜, 5.6˜)

$$\underline{c} = \underline{a}\,\underline{b} \Leftrightarrow \underline{c} = (\tilde{}\ c_{-M-N}\tilde{}\ldots\tilde{}\ c_0\tilde{},\ c_1\tilde{}\ldots\tilde{}) \tag{3}$$

where $c_k = \sum_{n=-N}^{k+M} a_n b_{k-n}$, $k = -M-N,\ldots,0,1,\ldots$,

$\underline{a} = (\tilde{}\ a_{-N}\tilde{}\ldots\tilde{}\ a_0\tilde{},\ a_1\tilde{}\ldots\tilde{})$, $\underline{b} = (\tilde{}\ b_{-M}\tilde{}\ldots\tilde{}\ b_0\tilde{},\ b_1\tilde{}\ldots\tilde{})$.

The PNs with these definitions of addition and multiplication form a field. A neutral element of addition is (˜0˜) and $-\underline{a} = (\tilde{}\ -a_{-N}\tilde{}\ldots\tilde{}\ -a_0\tilde{},\ -a_1\tilde{}\ldots\tilde{})$. A neutral element of multiplication is (˜1˜). Assuming $a_{-N} \neq 0$

$$\underline{a}^{-1} = \underline{d} = (\tilde{}\ 0\tilde{}, 0\tilde{}\ldots\tilde{}\ 0\tilde{}\ d_N\tilde{}\ d_{N+1}\tilde{}\ldots\tilde{}) \qquad (4)$$

where

$$d_N = a_{-N}^{-1}\ ,\ \ d_{N+k} = -d_N \sum_{n=-N+1}^{k-N} d_{k-n} a_n\ ,\quad k = 1, 2, \ldots$$

A mapping $a_0 \leftrightarrow (\tilde{}\ a_0\tilde{})$ is an isomorphism of the "digits" field and the PN field. Therefore the PN field is an extension of the "digits" field and it is possible to write $(\tilde{}\ a_0\tilde{}) = a_0$.

Multiplication by (˜1˜0˜) shifts the "digits" of the PN left:

$$(\tilde{}1\tilde{}0\tilde{})\ (\tilde{}\ a_{-N}\tilde{}\ldots\tilde{}\ a_0\tilde{}, a_1\tilde{}\ a_2\ldots\tilde{}) = (\tilde{}\ a_{-N}\tilde{}\ldots\tilde{}\ a_0\tilde{}\ a_1\tilde{}, a_2\ldots\tilde{}) \qquad (5)$$

and any PN can be represented by a series:

$$\underline{a} = (\tilde{}\ a_{-N}\tilde{}\ldots\tilde{}\ a_0\tilde{}, a_1\tilde{}\ldots\tilde{}) = \sum_{n=-N}^{\infty} a_n\ (\tilde{}1\tilde{}0\tilde{})^{-n} \qquad (6)$$

therefore (˜1˜0˜) serves the role of a basis of the "numerical system". Any PN can be expressed in "scientific notation":

$$\underline{x} = (\tilde{}\ x_0\tilde{}, x_1\tilde{}\ x_2\tilde{}\ldots\tilde{})\ (\tilde{}1\tilde{}0\tilde{})^N \qquad (7)$$

For $\underline{x} \neq (\tilde{}0\tilde{})$ a normalised form of the PN can be defined - this is the form (7) with $x_0 \neq 0$. In this case, N will be called a rank of the PN. The rank for $\underline{x} = (\tilde{}0\tilde{})$ is undefined. The normalised form plays a special role in the computer implementation of the PN arithmetic - this implementation is analogous to the floating-point arithmetic. The PN is saved as a pair of a finite sequence and an integer number N :

$$(\{x_0, x_1, x_2, \ldots, x_M\}, N\ ) \qquad (8)$$

which corresponds to the PN

$$(\tilde{}\ x_0\tilde{}, x_1\tilde{}\ x_2\tilde{}\ldots\tilde{}x_M\tilde{})\ (\tilde{}1\tilde{}0\tilde{})^N \qquad (9)$$

There are 3 simplifications in this representation. Firstly, N is an integer number from the limited set of value. Secondly, PN "digits" sequence is cut to the finite number of elements. Thirdly, each "digit" (for example, real number) has its own approximate representation.

The PN multiplication (3) is based on the digital convolution, which can be realised using FFT algorithm. Other operation such as inversion and power can be expressed using multiplication. A "PN co-processor" based on the hardware FFT processor can be used.

2.3. Functions

In further considerations it will be assumed, that the "digits" of the PNs are real numbers.

A convergence in the PN field can be entered using a concept of partially ordered Mikusinski space [3]. A set of non-negative PN and an absolute value of the PN can be defined by formulae:

$$\underline{a} \geq (\tilde{}0\tilde{}) \Leftrightarrow a_n \geq 0,\ n = -N, \ldots, 0, 1, \ldots$$
$$|\ \underline{a}\ | = (\tilde{}\ |a_{-N}|\tilde{}\ldots\tilde{}\ |a_0|\tilde{}, |a_1|\tilde{}\ldots\tilde{}) \qquad (10)$$

where $\underline{a} = (\tilde{}\ a_{-N}\tilde{}\ldots\tilde{}\ a_0\tilde{}, a_1\tilde{}\ldots\tilde{})$. A sequence of PN $\{{}_n\underline{x}\}$ will be called convergent when

$$\{{}_n\underline{x}\} \rightarrow \underline{x} \Leftrightarrow \qquad (11)$$
$$\Leftrightarrow \exists\, \underline{f} \geq (\tilde{}0\tilde{})\ \forall \varepsilon > 0\ \exists\, N\ \forall n{>}N\quad |\ {}_n\underline{x} - \underline{x}\ | \leq \varepsilon\, \underline{f}$$

Index n concerns the sequence element, which is the PN, and is placed on the left side to distinguish it from the index of the PN "digits".

Some useful functions can be defined as a sum of the series. For example:

$$e^{\underline{x}} = \sum_{n=0}^{\infty} \frac{\underline{x}^n}{n!} \qquad (12)$$

It can be proved [7] that the sequence of the partial sum of this series is always divergent, when rank of $\underline{x}$ is positive, and is convergent, when rank is non-positive. This statement corresponds to remark of Mikusinski [9, 10], and is incompatible with the statement of Bellert [1]. A "digits" of the function result for a given argument can be determined explicitly ([13] - Taylor coefficients of function):

$$\underline{x} = (\tilde{}\ x_0\tilde{}, x_1\tilde{}\ x_2\tilde{}\ldots\tilde{}),$$
$$\underline{y} = (\tilde{}\ y_0\tilde{}, y_1\tilde{}\ y_2\tilde{}\ldots\tilde{}),\ n = 1, 2, 3, \ldots$$
$$\underline{y} = \exp(\underline{x}) \Rightarrow y_0 = \exp(x_0),\ y_n = \sum_{k=1}^{n} \frac{k}{n} y_{n-k}\, x_k \qquad (13)$$
$$\underline{x} = \ln(\underline{y}) \Rightarrow x_0 = \ln(y_0),\ x_n = \frac{1}{y_0}\Big(y_n - \sum_{k=1}^{n-1} \frac{k}{n} y_{n-k}\, x_k\Big)$$
$$\underline{c} = \cos(\underline{x}),\ \underline{s} = \sin(\underline{x}) \Rightarrow c_0 = \cos(x_0),\ s_0 = \sin(x_0),$$
$$c_n = \sum_{k=1}^{n} \frac{k}{n} s_{n-k}\, x_k\ ,\ \ s_n = \sum_{k=1}^{n} \frac{k}{n} c_{n-k}\, x_k$$
$$\underline{y} = \underline{x}^a \Rightarrow y_0 = x_0^{\,a},\ \ y_n = \frac{1}{x_0} \sum_{k=1}^{n} \frac{(a+1)k - n}{n} y_{n-k}\, x_k$$

where a is a real number, in peculiarities a = -1 (inversion), a = 0.5 (square root).

3. OPERATIONAL CALCULUS

A definition of an operational calculus is based on Bittner approach [2, 3, 4]

3.1. Derivative, integral, constants

Let L^0 and L^1 are linear spaces (of functions, of series etc.), S, T, s are linear operations, and

$$S: L^1 \mapsto L^0,\ S(L^1)=L^0$$
$$T: L^0 \mapsto L^1$$
$$s: L^1 \mapsto L^1 \tag{14}$$

$$S\,T\,f = f,\ f \in L^0$$
$$T\,S\,x = x - s\,x,\ x \in L^1$$

Quintuple (L^0,L^1,S,T,s) will be called an operational calculus with the derivative S, the integral T, and the limit condition s and will be denoted $\mathrm{CO}(L^0,L^1,S,T,s)$.

Example 3.1.

$L^0 = C^0(a,b)$, $L^1 = C^1(a,b)$, $S = \dfrac{d}{dt}$, $T = \int_{t_0}^{t}$, $s = |_{t_0}$, i.e.

S $\{x(t)\} = \{ \frac{d}{dt} x(t) \}$, $T\,\{f(t)\} = \{ \int_{t_0}^{t} f(\tau)d\tau \}$,

$s\,\{x(t)\} = \{x(t_0)\}$, where $t_0 \in (a,b)$.

As in [9, 10], $\{x(t)\}$ denotes function x, and is used to distinguish it from $x(t)$ - value of the function x at the point t. The limit condition s maps the function $\{x(t)\}$ to the function $\{x(t_0)\}$ which has constant value $x(t_0)$ for all arguments from (a,b).

Example 3.2.

$L^0 = L^1 = C(N)$ - space of numerical sequences $\{x_k\}$,

$S\,\{x_0,x_1,x_2,...\} = \{x_1,x_2,x_3,...\}$,

$T\,\{x_0,x_1,x_2,...\} = \{0,x_0,x_1,...\}$,

$s\,\{x_0,x_1,x_2,...\} = \{x_0,0,0,0,...\}$

3.2. Taylor formula

When $L^1 \subset L^0$, it is possible to introduce the n-th iteration of the integral: $T^n f = T(T...(T(Tf))...)$ and the n-th iteration of the derivative: $S^n x = S(S...(S(Sx))...)$ where $S^n: L^n \mapsto L^0$, $L^n \subset L^0$.

$x \in L^n$ can be expressed by the Taylor formula [3, 4]:

$$x = x_0 + Tx_1 + T^2x_2 + ... + T^{n-1}x_{n-1} + T^nS^nx \tag{15}$$

where $x_k = sS^kx$, $k = 0, 1, ..., n$-1 .

3.3. Space of results, operators

Let us consider the equation:

$$U\,x = f \tag{16}$$

where f belongs to an arbitrary space X, U is a linear operation and $U(X) \subset X$. It is possible that solution x not always exists in X. But when U satisfies some conditions (see [4, 3, 14]), the solution can be "found" (is unique) in a space of "fractions" $\dfrac{f}{U}$, which will be called the space of results $\Xi(X)$. This space of results is an extension of X through isomorphism: $f \mapsto \dfrac{f}{I}$, where I is identity operation.

Example 3.3.

$X = C^1(0,\infty)$, $U = \int_0^t$. A solution x of the equation $U\,x = \{1\}$ does not exist in X, but exists in $\Xi(X)$. $\{1\}$ denotes Heaviside function with the value 1 for all $t \in (0,\infty)$.

Similarly, the "fractions" $\dfrac{A}{U}$ can be considered, where A and U are operations. These fractions will be called operators over the space of results $\Xi(X)$. A *Heaviside operator* will be defined as

$$p := \frac{I}{T} \tag{17}$$

where I is an identity operation and T is the integral of the operational calculus. Taylor formula (15) can be expressed using Heaviside operator:

$$S^nx = p^nx - p^nx_0 - p^{n-1}x_1 - ... - p\,x_{n-1} \tag{18}$$

where $x_k = s\,S^kx$, k=0, 1, ..., n-1. This formula can be used to find a solution of a differential equation as a function of p.

4. LAPLACE AND Z TRANSFORM

In the operational calculus a concept of transform plays a less significant role, because it is not necessary to distinguish space of functions and space of operators (transforms). For example, the function $\{t\}$ can be expressed in many equivalent forms:

$$\{t\} = T\,\{1\} = \frac{I}{p}\{1\} = \frac{I}{p^2}\frac{\{1\}}{T} = ... \tag{19}$$

where

$$L^0 = C^0(0,\infty),\ L^1 = C^1(0,\infty),\ S = \frac{d}{dt},\ T = \int_0^t,\ s = |_0 \tag{20}$$

However, by fixing one element $e \in \Xi(L^0)$ from the results space, it is possible to represent any element $a \in \Xi(L^0)$ through the expression:

$$a = A\,e \tag{21}$$

where A is an operator over the space of results $\Xi(L^0)$. This operator A can be treated as a transform of the element a (see [12]).

4.1. PN and transforms

Let us define a product of the PN (˜1˜0˜) and an arbitrary element of the result space $x \in \Xi(X)$:

$$(\tilde{}1\tilde{}0\tilde{})\,x := p\,x \tag{22}$$

and let us assume that the operation of addition and multiplication in the PN field corresponds to the operation of addition and multiplication (composition) of the operators. The PN corresponds to the operator expressed in the form of a series:

$$\underline{a} = (\tilde{}\ a_{-N}\tilde{}\ ...\tilde{}\ a_0\tilde{}, a_1\tilde{}\ ...\tilde{}) = \sum_{n=-N}^{\infty} a_n\, p^{-n} \tag{23}$$

4.2. Laplace - Carson transform

Let us consider CO(L^0, L^1, S, T, s) as defined in (20) and e = {1}. An operator A is Laplace-Carson transform of $a \in \Xi(L^0)$, which will be denoted as $A = \boldsymbol{L}_C(a)$, when satisfies formula:

$$a = A\, e = A\{1\} \tag{24}$$

For example, $\boldsymbol{L}_C(\{t\}) = p^{-1} = (\tilde{}\ 0\tilde{}, 1\tilde{})$.

4.3. Laplace transform

Let us consider CO(L^0, L^1, S, T, s) as defined in (20) and $e = \frac{\{1\}}{T} = p\ \{1\}$. An operator A is Laplace transform of $a \in \Xi(L^0)$, which will be denoted as $A = \boldsymbol{L}(a)$, when satisfies formula:

$$a = A\, e = A\ p\{1\} \tag{25}$$

For example, $\boldsymbol{L}(\{t\}) = p^{-2} = (\tilde{}\ 0\tilde{}, 0\tilde{}\ 1\tilde{})$. Heaviside operator p corresponds here to the variable s of the Laplace transform - a complex function.

4.4. Z transform

Let us consider CO(L^0, L^1, S, T, s), where $L^0 = L^1$ is space of numerical series $\{x_0,x_1,x_2,...\}$,

$$\begin{aligned} &S\ \{x_0,x_1,x_2,...\} = \{x_1,x_2,x_3,...\} \\ &T\ \{x_0,x_1,x_2,...\} = \{0,x_0,x_1,...\} \\ &s\ \{x_0,x_1,x_2,...\} = \{x_0,0,0,0,...\} \\ &e = \{1,0,0,0,...\} \end{aligned} \tag{26}$$

An operator A is Z transform of $a \in \Xi(X)$, what will be denoted as $A = \boldsymbol{Z}(a)$, when satisfies formula:

$$a = A\, e = A\ \{1,0,0,0,...\} \tag{27}$$

Heaviside operator $1/T$ corresponds here to the variable z of the Z transform - a complex function; therefore, we will denote it by z in this paper. For example, $\boldsymbol{Z}(\{1,1,1,...\}) = (\tilde{}\ 1\tilde{}, 1\tilde{}\ 1\tilde{}...\tilde{}) = z^0 + z^{-1} + z^{-2} + ... = \frac{1}{1-z^{-1}} = \frac{1}{(\tilde{}1\tilde{}, -1\tilde{})}$. A PN, which is Z transform, has "digits" equal to the sequence elements:

$$\boldsymbol{Z}(\{x_0, x_1, x_2, ...\}) = (\tilde{}\ x_0\tilde{}, x_1\tilde{}\ x_2\tilde{}...\tilde{}x_M\tilde{}) \tag{28}$$

4.5. Inverse Laplace transform

Let $\underline{F} = (\tilde{}\ 0\tilde{}, F_1\tilde{}\ F_2\tilde{}\ F_3\tilde{}...\tilde{}) = \sum_{n=1}^{\infty} F_n\, p^{-n}$ be an Laplace transform (25) of the function $\{f(t)\}$. Inverse transform of $\underline{F}$ is the function:

$$\{f(t)\} = \underline{F}\ p\{1\} = \left(\sum_{n=1}^{\infty} F_n\ p^{-n}\right) p\{1\} = \sum_{n=1}^{\infty} F_n\ T^{n-1}\{1\}$$

$$= F_1\{1\} + F_2\{\frac{t^1}{1!}\} + F_3\{\frac{t^2}{2!}\} + ... \tag{29}$$

Computer implementation of the PN arithmetic limits time t for which we can obtain useful results due to two reason: series of "digits" F_n is cut, so we have a finite number of series components and each component is calculated by using a floating point arithmetic with a given number of bits. The maximal absolute value of the elements of the sum (29) and value of the last element can be used to determine accuracy of the result.

4.6. Laplace transform and discontinues functions

Discontinuous functions can be expressed using delay operator $\exp(-t_0\ p)$. Value of $\exp(-t_0\ p) = \exp(-t_0\ (\tilde{}1\tilde{}0\tilde{}))$ cannot be calculated using (12), because of a positive rank of argument - series is divergent. However, calculation for each delay time can be performed separately, and inverse transform can be determined as a sum of delayed components in the time domain. It is possible to introduce rules of 4 basic operations:

$$\underline{A}_1\exp((\tilde{}t_1\tilde{}0\tilde{})) \pm \underline{A}_2\exp((\tilde{}t_2\tilde{}0\tilde{})) = \begin{cases} (\underline{A}_1 \pm \underline{A}_2)\exp((\tilde{}\ t_1\tilde{}\ 0\tilde{}\ )) & \text{if } t_1 = t_2 \\ 2 \text{ summand with different delay time} & \text{if } t_1 \neq t_2 \end{cases}$$

$$\underline{A}_1\exp((\tilde{}t_1\tilde{}0\tilde{})) \cdot \underline{A}_2\exp((\tilde{}t_2\tilde{}0\tilde{})) = \underline{A}_1\cdot\underline{A}_2\exp((\tilde{}t_1 + t_2\tilde{}0\tilde{})) \tag{30}$$

$$\underline{A}_1\exp((\tilde{}t_1\tilde{}0\tilde{}))\ /\ (\underline{A}_2\exp((\tilde{}t_2\tilde{}0\tilde{}))) = (\underline{A}_1/\underline{A}_2)\ \exp((\tilde{}t_1 - t_2\tilde{}0\tilde{}))$$

In the computer implementation we will have a set of pairs: real number and PN (t_i , $\underline{A}_i$). New elements of the set can appear after an operation of addition. In the time domain each inverse Laplace transform of $\underline{A}_i$ will be added to the global solution, considering proper delay time t_i.

Discontinuous functions can also appear as a solution of a system described by the partial differential equation. For example, Laplace transform of the voltage at the point x of the long line with length l , series impedance per unit length $\underline{Z}_0 = R_0 + pL_0$ and shunt admittance per unit length $\underline{Y}_0 = G_0 + pC_0$ can be expressed by formula:

$$\underline{U}(x) = \frac{\underline{E}_1(1-\underline{\rho}_1)}{2}\ \frac{\exp(-\underline{\gamma}x) + \underline{\rho}_2\exp(\underline{\gamma}(x-2l))}{1-\underline{\rho}_1\underline{\rho}_2\exp(-2\underline{\gamma}l)} \tag{31}$$

where $\underline{\gamma} = \sqrt{\underline{Z}_0\underline{Y}_0} = (\tilde{}\ \gamma_{-1}\tilde{}\ \gamma_0\tilde{}, \gamma_1\tilde{}\ \gamma_2\tilde{}\ ...\tilde{})$ (see the algorithm $\underline{x}^a$ (13)), $\gamma_{-1}\ l = \tau = \sqrt{L_0C_0}$.

The terms of shape $\exp(\underline{\gamma}\alpha)$, where α = -x, or x-2l, or -2l are not convergent, because of the positive rank of γ, but we can separate a delay operator (see [11]):

$$\exp(\underline{\gamma}\alpha) = \exp((\tilde{}\,\tau\,\tilde{}\,0\,\tilde{}\,)) \cdot \exp(\tilde{}\,\alpha\gamma_0\,\tilde{}\,, \alpha\gamma_1\,\tilde{}\,\alpha\gamma_2\,\tilde{}\,...\,\tilde{}\,)$$
$$= \exp((\tilde{}\,\tau\,\tilde{}\,0\,\tilde{}\,)) \cdot \underline{A} \qquad (32)$$

where $\underline{A}$ is calculated by using the algorithm exp( ) (13). Further calculation can be performed by using a set of summand (30).

4.7. Laplace transform and partial differential equation with constant coefficients

Mikusinski proved [11], that the solution of a partial differential equation with constant coefficients of m order has a solution with exponential terms, which arguments are power series of $\sqrt[q]{p}$, where $q \le m$. For this case, mapping (˜1˜0˜) = p is useless, but mapping (˜1˜0˜) = $\sqrt[q]{p}$ should be used. For example, in the analysis of a long line considering skin effect series impedance per unit length contains Bessel functions of argument $\sqrt{p}$. Therefore, when we assume

$$(\tilde{}1\tilde{}0\tilde{}) = \sqrt{p} \quad (\text{so } p = (\tilde{}1\tilde{}0\tilde{}0\tilde{})) \qquad (33)$$

it is possible to use series expansion of Bessel function. By this mapping γ has form:

$$\gamma = (\tilde{}\gamma_{-2}\tilde{}\gamma_{-1}\tilde{}\ \gamma_0\tilde{},\ \gamma_1\tilde{}\ \gamma_2\tilde{}\ ...\tilde{}) \qquad (34)$$

As in [11], an exponential function can be presented in the form (where α is a real number):

$$\exp(\underline{\gamma}\alpha) = \exp((\tilde{}\,\alpha\gamma_{-2}\,\tilde{}\,0\,\tilde{}\,0\,\tilde{}\,)) \cdot \exp((\tilde{}\,\alpha\gamma_{-1}\,\tilde{}\,0\,\tilde{}\,)) \cdot \exp(\tilde{}\,\alpha\gamma_0\,\tilde{}\,, \alpha\gamma_1\,\tilde{}\,\alpha\gamma_2\,\tilde{}\,...\,\tilde{}\,) \qquad (35)$$

The term $\exp((\tilde{}\,\alpha\gamma_{-2}\,\tilde{}\,0\,\tilde{}\,0\,\tilde{}\,)) = \exp(\alpha\gamma_{-2}\,p)$ in (35) represents the delay operator. The term $\exp(\tilde{}\,\alpha\gamma_0\,\tilde{}\,, \alpha\gamma_1\,\tilde{}\,...\,\tilde{}\,)$ can be calculated using (13). The term $\exp((\tilde{}\,\alpha\gamma_{-1}\,\tilde{}\,0\,\tilde{}\,))$ is an exponential function with argument of the positive rank, therefore cannot be calculated, but an inverse Laplace transform of

$$\exp(-\beta\sqrt{p})\sum_{n=1}^{\infty} A_n \sqrt{p}^{\,-n} \qquad (36)$$

can be expressed as

$$\sum_{n=1}^{\infty} A_n y_n(t,\beta) \qquad (37)$$

where

$$y_0(t,\beta) = \frac{1}{\sqrt{\pi t}}\exp\left(\frac{-\beta^2}{4t}\right),\ y_1(t,\beta) = \operatorname{cerf}\left(\frac{\beta}{2\sqrt{t}}\right),$$
$$y_{n+1}(t,\beta) = \frac{2t\,y_{n-1}(t,\beta) - \beta\,y_n(t,\beta)}{n}$$

In the computer implementation we will have a set of triple - 2 real numbers and PN: (t_i , β_i , $\underline{A}_i$) corresponds to summand:

$$\exp((\tilde{}\,t_i\,\tilde{}\,0\,\tilde{}\,0\,\tilde{}\,)) \cdot \exp((\tilde{}\,\beta_i\,\tilde{}\,0\,\tilde{}\,)) \cdot \underline{A}_i$$

The solutions of the long-line equations with Bessel functions are shown in [6].

4.8. Z transform and solution of differential equation

Z transform is suitable to solve difference equation. Inverse Z transform in the PN field is trivial - "digits" of the PN are equal to the sequence elements (28). The Z transform can also be utilised to find an approximate solution of the differential equation. Let us consider a Laplace transform $X(p)$ of a function $\{x(t)\}$. According to (25) it satisfies a formula:

$$\{x(t)\} = X(p)\ p\ \{1\} \qquad (38)$$

The Laplace transform $X(p)$ as a function of p can be derived form difference equation, applying (18). Substituting constant function {1} by a series of constant samples {0.5, 1, 1, 1, ...} (with the value 0.5 in a discontinuous point of function, see [7]) and the operator $p = \frac{1}{\int_0^t}$ by its discrete equivalent, we get a formula for the approximate series of samples of function x :

$$\underline{x} = X(\underline{p})\ \underline{p}\ (\tilde{}\,0.5\,\tilde{}\,, 1\,\tilde{}\,1\,\tilde{}\,1\,\tilde{}\,...\,\tilde{}\,) \qquad (39)$$

where $\underline{p}$ is the fixed polynomial number corresponding to the algorithm of the numerical integration. For example, when the trapezoidal method is considered

$$\underline{p} = \frac{2}{h}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{h}\,\frac{(\tilde{}\,1\,\tilde{}\,,-1\,\tilde{}\,)}{(\tilde{}\,1\,\tilde{}\,,1\,\tilde{}\,)} \qquad (40)$$

where h is the sampling period. "Digits" of the polynomial number $\underline{x}$ are approximately equal to samples of the function $\{x(t)\}$:

$$\underline{x} \cong (\tilde{}\,x(0)\,\tilde{}\,, x(1h)\,\tilde{}\,x(2h)\,\tilde{}\,x(3h)\,\tilde{}\,...\,\tilde{}\,) \qquad (41)$$

Z transform as the PN in the case of models described by rational transfer function has not advantages in comparison to the classical discrete approach, but can be used to transform formulas from AC signal domain to the transient domain. In [8] is presented an algorithm in which Z transform as the PN is used to calculate time domain characteristics of a large circuit where some parameters of the circuit are variable. This approach is based on a semi-symbolic method developed for the AC signals [5]. For example, computational time for the circuit with 8 operational amplifiers, 20 resistors and 4 capacitors was 4.2ms per one characteristic (Pentium 130MHz). It is short enough to build a simulator with "animated graph", where characteristics are generated in accordance with a position of the mouse pointer over plane of the parameters.

4.9. Z transform and non-rational transfer function

All arguments of PN functions in Z transform case are PNs of rank at the most 0; therefore, all functions can be calculated using algorithms (13). Let us consider as an example a Laplace transform in the form:

$$F(p) = \frac{1}{p^2 + p + 4}\exp(-4\sqrt{p^2+1}) \qquad (42)$$

A graph of an inverse Laplace obtained by using (32) and (29) is shown in fig. 1 as a solid line.

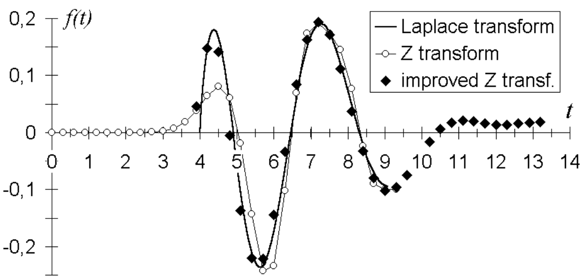

Fig. 1. Z transform corresponds to a non-rational Laplace transform.

Applying (39) and (40) with (42) we can get an approximate solution shown in fig. 1 as circles. To make errors more visible, all calculations were made with a small number of "digits" of the PN and with relatively high sampling step *h*.

A delay operator can be easily expressed in Z transform, because it denotes shifting of the sequence. In the PN field, it corresponds to the multiplication by $(\tilde{}1\tilde{}0\tilde{})^{-N}$, where *Nh* is delay time. We can separate delay from (42), and represent Z transform in a form:

$$\underline{f} = F(\underline{p})\,\underline{p}\,(\tilde{}0.5\tilde{},1\tilde{}1\tilde{}\ldots\tilde{}) = \qquad (43)$$
$$= (\tilde{}1\tilde{}0\tilde{})^{-N}\frac{1}{\underline{p}^2 + \underline{p} + 4}\exp(-(4\sqrt{\underline{p}^2+1} - Nh\underline{p}))$$

where $N = \left\lfloor \frac{4}{h} \right\rfloor$, $\lfloor\ \rfloor$- integer part of the number. The digits of $\underline{f}$ calculated according to (43) are shown in fig. 1 as rhombuses.

5. CONCLUSION

The polynomial number method refers to the computer implementation of some operational calculi and is analogous to the floating-point arithmetic. The main advantage of the PN method is unified numerical representation of transient signals; therefore, formulas and algorithms used for AC signals can be efficiently applied in the transient domain. The main disadvantage is that time period of any solution is restricted in advance, due to fixed in advance number of PN "digits". A special field of utilisation of the PN method is systems described by non-rational transfer functions.

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